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ON THE LIMITING STRENGTH OF ROCKS UNDER CONDITIONS OF STRESS EXISTING IN THE EARTH'S INTERIOR

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§ 1. INTRODUCTION

One of the most important problems in geophysics is to obtain a knowledge of the limiting strength of the material forming the solid core of the earth as well as that of the principal rocks forming its crust. According to the modern theory of elasticity the tendency of a solid to rupture is measured by the maximum difference of the greatest and least principal stresses. This criterion first due to Tresca and followed by Sir G. H. Darwin[1] has been found by J. J. Guest[2] to give the best agreement with observed results from experiments on metal tubes subjected to various systems of combined stress. Coulomb's suggestion that the greatest shear produced in a material is a measure of its tendency to rupture leads to a result substantially the same as that given by the hypothesis of maximum stress-difference. This follows from the theorem[3] that the shearing stress at any point is greatest on a plane whose normal lies in the plane of the axes of the algebraically greatest and algebraically least stress and bisects the angle between them, its value being half the algebraic difference between these stress-intensities. The family of surfaces whose tangent planes satisfy the above condition determines the surfaces along which flaws will develop or the material commence to rupture. These surfaces are are called *surfaces of maximum shear.*

The difficulty in the way of the practical application of these criteria lies especially in the determination of the numerical values of the limiting stress-difference for different materials. For the purposes of ordinary engineering practice, a cube of the material is put into a testing-machine and the stress required to crush the cube is measured. The value of this stress per unit area gives the value of the limiting stress-difference for the material *as long as the boundary conditions are similar to those existing during the test.* Although this condition generally holds in engineering construction work and the criterion is valid, this may by no means be the case when the boundary conditions are very different from those existing in the test, as for instance in the case of rupture in the neighborhood of a cavity or in the case of rupture due to stress-difference deep down in the earth's crust. It has been customary up to the present to employ the crushing test criterion in applications to stresses in the earth's interior.[1] A mathematical analysis of the experiments of Dr. Adams,[2] described in the accompanying paper, throws a great deal of light on the question and can be made to give valuable data on the limiting strength of rocks under conditions of extreme pressure.

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[1] Darwin, "On the Stresses Produced in the Interior of the Earth by the Weight of Continents and Mountains," *Phil. Trans. Roy. Soc.*, CLXXIII (1882).

[2] Guest, *Phil. Mag.* (Ser. 5), XLVIII (1900).

[3] Minchin, *Statics*, 4th ed., 1889, II, 425.

It will appear that the crushing test criterion of limiting stress-difference (e.g., 27,000 lbs. per sq. in. for a cubical specimen of Westerly granite of two inches side) is much too low when conditions of stress approach those existing in the earth's interior.

For the purposes of geodynamics, however, "we require to know what is the limiting stress-difference under which a material takes a permanent set or begins to flow rather than the stress-difference under which it breaks; for if the materials of the earth were to begin to flow, the continents would sink down and the sea-bottoms rise up."[3]

A mathematical analysis of tests described in § 4 afford a valuable means of obtaining information on this question.

§ 2. CONDITIONS OF TEST

The tests described by Dr. Adams on the compression of cylindrical rock specimens containing cylindrical cavities and inclosed in heavy nickel-steel jackets lend themselves especially well to a determination of stress-differences according to the elastic solid theory. An indeterminate condition which might vitiate the application of analysis to the problem is the state of initial stress brought about by shrinking the nickel-steel jacket over the rock specimen. Since the ends of the specimen are free during this process it is probable that the rock will adapt itself to a state of small initial stress by slight displacements in the direction of the axis. This will especially be the case if some time is allowed to elapse before the test is carried out. Care was taken to make this stress as small as possible under the circumstances; see p. 105.

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[1] Darwin, *loc. cit.*

[2] Adams, "An Experimental Contribution to the Question of the Depth of the Zone of Flow," see this journal, p. 97.

[3] Darwin, *loc. cit.*

One of the most serious objections to the application of the theory of elasticity to the determination of conditions of rupture lies in the fact that in most tests the conditions of small strains upon which the theory is based are violated long before rupture takes place. Under the conditions just described the strains remain small in consequence of the very slight yielding of the nickel-steel jacket. It is well known that the elastic constants are only determinate so long as the strains are small and that when the material is stressed beyond a certain limit the rigidity and compressibility take quite different values, so that in most cases the values of the stresses just before rupture cannot be calculated from the conditions of the test. In the present instance, however, by measuring the lateral dilatation of the nickel-steel jacket during a test in the manner described in §4, it is possible to determine how the elastic constants of the rock change in value with stress. In this way it is legitimate to make use of the equations of the elastic solid theory, provided we employ *instantaneous* values of the constants for the specimen and as long as the stresses in the nickel steel do not exceed the limiting stress for that material.

§ 3. DETERMINATION OF STRESSES IN ROCK CYLINDER UNDER TEST

If we consider the state of stress in the rock specimen and in the nickel-steel jacket as one of plane stress, the problem may be solved without an undue degree of complication. The measurements given in the next section of the lateral dilatation of the nickel-steel jacket when the rock specimen is under pressure show a considerable bulge of the equatorial diameter, showing that the problem is not strictly one of plane stress. The complete solution of the problem would be of the nature of a similar one considered by Filon[1] in which the state of stress is determined for a cylinder pressed between planes which are in contact with its ends. For the purposes of the present paper, however, we shall deal with the problem as one of plane stress and neglect surface tractions, over the boundary of the cylindrical specimen due to relative displacements of the rock surface and the steel jacket when under pressure. The results will represent to a first approximation the state of stress existing in the rock specimen. From the general solution of the problem given by Love[2] it is an easy matter to calculate the stress-differences at any point of the specimen. According to the usual notation we denote by $\widehat{rr}$, $\widehat{\theta\theta}$, and $\widehat{zz}$ the radial, transverse, and axial stresses respectively: for a case of plane stress there are also the principal stresses. We denote the differences of the principal stresses by the notation

(i) $\widehat{rr}-\widehat{zz}$ (ii) $\widehat{rr}-\widehat{\theta\theta}$ (iii) $\widehat{\theta\theta}-\widehat{zz}$

From the theorem cited in § 1 it is seen that each of the stress-differences (i), (ii), (iii) is associated with a family of surfaces of maximum shear along which the material will crack or flow.

(i) The surfaces of maximum principal shear due to the stress-differences $(\widehat{rr}-\widehat{zz})$ consist of two systems of cones of semi-vertical angle 45° and cutting each other orthogonally. One such system may be imagined to consist of a pile of glass funnels fitting into one another.

(ii) The surfaces of shear due to the stress-differences $(\widehat{rr}-\widehat{\theta\theta})$ consist of two systems of mutually orthogonal cylindrical surfaces whose traces on a plane perpendicular to the axis of the cylinder are equiangular spirals cutting the radii at angles of 45°.

(iii) The stress-differences $(\widehat{\theta\theta}-\widehat{zz})$ give rise to two families of helicoidal surfaces of pitch 45° and intersecting orthogonally. The traces of these surfaces on the surface of the cylinder give rise to helices well known as Luder's lines.

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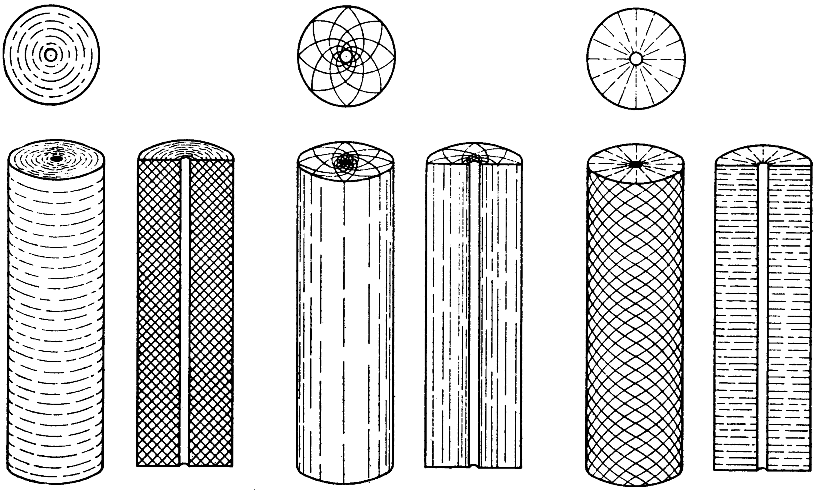
[1] L. N. G. Filon, "On the Elastic Equilibrium of Circular Cylinders under Certain Practical Systems of Load," *Phil. Trans. Roy. Soc.*, CXCVIII A, 1902, 182.

[2] Love, *Elasticity*, 2d ed., 140.

The systems of surfaces of shear corresponding to each of the above cases are shown as (i), (ii), and (iii) in Fig. 1. The cracks developed by failure in each of the above ways as they would appear on the cylindrical surface, over a cross-section perpendicular to the axis and over a section through the axis are represented by the diagrams (i), (ii), and (iii).



(i) (ii) (iii)

FIG. 1

The surfaces of shear developed in a cylinder under test will depend on which of the stress-differences is a maximum, a condition determined by the elastic constants of the rock and also by the nature of the nickel-steel boundary. As soon as this maximum stress-difference has attained a limiting value characteristic of the material, and determined to some extent by the boundary conditions the corresponding families of surfaces of shear will make their appearance.

Considering first the stresses in the rock specimen, the radial displacement U satisfies the equation

$$\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right)=0,$$

of which the complete primitive is

$$U = Ar + \frac{B}{r}, \tag{1}$$

A and B being constants to be determined from the specified boundary conditions. Longitudinal stress superimposed upon the radial stresses can be taken into account by taking the displacement w in the direction of the axis to be $w = ez$, e being a constant representing a uniform longitudinal extension.

If we denote by $\widehat{rr}$ the stress-component along the radius r, by $\widehat{\theta\theta}$ the component at right angles to r and by $\widehat{zz}$ that along the axis, we find from the stress-equations of equilibrium,

$$\left.\begin{aligned}
\widehat{rr} &= (\lambda + 2\mu)\frac{\partial U}{\partial r} + \lambda\frac{U}{r} + \lambda\frac{\partial w}{\partial z} \\
\widehat{\theta\theta} &= (\lambda + 2\mu)\frac{U}{r} + \lambda\frac{\partial U}{\partial r} + \lambda\frac{\partial w}{\partial z} \\
\widehat{zz} &= (\lambda + 2\mu)\frac{\partial w}{\partial z} + \lambda\left(\frac{U}{r} + \frac{\partial U}{\partial r}\right)
\end{aligned}\right\} \tag{2}$$

According to Lamé's notation μ is the modulus of rigidity and λ one of the moduli of elasticity such that $\kappa = (\lambda + \frac{2}{3}\mu)$ is the modulus of compression. Poisson's ratio, σ, is denoted by $\sigma = \dfrac{\lambda}{2(\lambda + \mu)}$.

In terms of A and B of equation (1) we have for the principal stresses and the hydrostatic pressure p

$$\left.\begin{aligned}
\widehat{rr} &= 2(\lambda + \mu)A - \frac{2\mu}{r^2}B + \lambda e \\
\widehat{\theta\theta} &= 2(\lambda + \mu)A + \frac{2\mu}{r^2}B + \lambda e \\
\widehat{zz} &= 2\lambda A + (\lambda + 2\mu)e \\
p &= -\tfrac{1}{3}(\widehat{rr} + \widehat{\theta\theta} + \widehat{zz}) = -\kappa(A + e)
\end{aligned}\right\} \tag{3}$$

If accented letters refer to corresponding quantities in the coaxial nickel-steel jacket, we have for the principal stresses a set of equations analogous to (1) and (3),

$$\left.\begin{aligned}
U' &= A'r' + B'/r' \\
\widehat{rr}' &= 2(\lambda'+\mu')A' - \frac{2\mu'}{r'^2}B' + \lambda' e' \\
\widehat{\theta\theta}' &= 2(\lambda'+\mu')A' + \frac{2\mu'}{r'^2}B' + \lambda' e' \\
\widehat{zz}' &= 2\lambda' A' + (\lambda'+2\mu')e'
\end{aligned}\right\} \tag{4}$$

Let a be the radius of the cylindrical cavity in the rock specimen, b the radius of the specimen, and c the radius of the nickel-steel jacket. The boundary conditions give six equations to determine the constants A, B, e, A', B', e'.

At $r=a$, $\widehat{rr}=0$, giving

$$2(\lambda+\mu)A - 2\mu B/a^2 + \lambda e = 0 \tag{5}$$

At $r=r'=b$, $\widehat{rr}=\widehat{rr}'$, giving

$$2(\lambda+\mu)A - 2\mu B/b^2 + \lambda e = 2(\lambda'+\mu')A' - 2\mu' B'/b^2 + \lambda' e' \tag{6}$$

At $r=r'=b$, $U=U'$, giving

$$Ab + B/b = A'b + B'/b \tag{7}$$

At $r'=c$, $\widehat{rr}=0$, giving

$$2(\lambda'+\mu')A' - 2\mu' B'/c^2 + \lambda' e' = 0 \tag{8}$$

In addition to these we have in the rock specimen, $\widehat{zz}=-P$, where P is the pressure per unit area applied to the end of the test cylinder; this condition gives

$$2\lambda A + (\lambda+2\mu)e = -P \tag{9}$$

Also since the nickel-steel jacket is free from longitudinal stress,

$$\widehat{zz}' = 0 \text{ or } 2\lambda' A' + (\lambda'+2\mu')e' = 0 \tag{10}$$

We denote by β the quantity,

$$\beta = \frac{1+\sigma}{1-\sigma}\cdot\frac{\mu}{\mu'}\left\{\left(1+\frac{1-\sigma'}{1+\sigma'}\;\frac{b^2}{c^2}\right)\frac{1-a^2/b^2}{1-b^2/c^2} + \frac{\mu'}{\mu}a^2/b^2\right\} \tag{11}$$

We then find from the six equations (5) to (10),

$$\left.\begin{aligned}
\frac{B}{a^2} &= -\frac{P}{2\mu}\cdot\frac{\sigma}{1-\sigma}\;\frac{1}{1+\beta} \\
&\text{and} \\
e - A &= -\frac{P}{\lambda+2\mu}\left\{1+\frac{\sigma}{1-2\sigma}\;\frac{\beta}{1+\beta}\right\}
\end{aligned}\right\} \tag{12}$$

The principal stress-differences in the rock specimen are from (3)

$$\left.\begin{aligned}
&\text{(i)}\ \widehat{rr}-\widehat{zz}=-2\mu(e-A+B/r^2)\\
&\qquad\qquad=\frac{\sigma}{1-\sigma}P\left\{\frac{1-2\sigma}{\sigma}+\frac{\beta}{1+\beta}+\frac{1}{1+\beta}\,\frac{a^2}{r^2}\right\}\\
&\text{(ii)}\ \widehat{rr}-\widehat{\theta\theta}=-4\mu B/r^2\quad=\frac{\sigma}{1-\sigma}P\left\{\frac{2}{1+\beta}\,\frac{a^2}{r^2}\right\}\\
&\text{(iii)}\ \widehat{\theta\theta}-\widehat{zz}=-2\mu(e-A-B/r^2)\\
&\qquad\qquad=\frac{\sigma}{1-\sigma}P\left\{\frac{1-2\sigma}{\sigma}+\frac{\beta}{1+\beta}-\frac{1}{1+\beta}\,\frac{a^2}{r^2}\right\}
\end{aligned}\right\}\qquad(13)$$

The radial displacement U at the outer surface of the rock specimen is given by

$$\frac{U}{b}=\frac{P}{2\mu}\,\frac{\sigma}{1+\sigma}\,\frac{\beta-\dfrac{a^2}{b^2}\,\dfrac{1+\sigma}{1-\sigma}}{1+\beta}\qquad(14)$$

and the longitudinal extension per unit length is given by

$$e=-\frac{P}{2\mu}\,\frac{1}{1+\sigma}\,\frac{\beta+\dfrac{(1+\sigma)(1-2\sigma)}{1-\sigma}}{1+\beta}\qquad(15)$$

The radial displacement U' at the outer surface of the nickel-steel jacket is given by

$$\frac{U'}{c}=\frac{P}{\mu'}\,\frac{b^2}{c^2}\,\frac{1-a^2/b^2}{1-b^2/c^2}\,\frac{\sigma}{(1-\sigma)(1+\sigma')(1+\beta)}\qquad(16)$$

and the longitudinal extension per unit length is given by

$$e=-\sigma'\frac{U'}{c}\qquad(17)$$

§ 4. DISCUSSION OF EXPERIMENTAL RESULTS

In order to obtain a numerical verification of the distribution of stress in the rock specimens, measurements of the lateral bulging of the nickel-steel jacket were made by means of a very sensitive Ewing extensometer for a series of increasing loads. The results were obtained for a specimen of Solenhofen limestone in which an axial and transverse cavity had been drilled, and the experiments carried out in the Engineering Testing Laboratory of McGill University.

TABLE I

TABLE GIVING RADIAL DISPLACEMENT OF NICKEL-STEEL JACKET CONTAINING A SPECIMEN OF SOLENHOFEN LIMESTONE UNDER VARYING LOADS

P Pounds per Square Inch	$\frac{U'}{c}\times 10^{-6}$ Radial Displacement at Center	$\frac{U'}{c}\times 10^{-6}$ Radial Displacement over End	Remarks
5,100	0	0	Diameter of nickel-steel jacket 2.5 in., $c=1.25$; length of nickel-steel jacket 3.25 in.
25,500	16	0	
51,000	34	4	Diameter of rock specimen before loading .5005 in., $b=.2502$ in.; length of rock specimen before loading 1.5725 in.
76,500	50	10	
102,000	76	18	
127,000	106	24	*Load removed*, diam. of specimen at ends .5052 in.; diam. of specimen at equator .5030 in.; length 1.5437 in.
153,000	154	38	
175,000	210	46	
200,000	326	66	Diameter of cylindrical cavities .050 inch.
5,100	124	38	*Load removed*, vertical cavity filled with rock powder, horizontal cavity distorted.
1,000	124	38	
1,000	116	37.5	Longitudinal compression of jacket between 2 in. centers .00003 in., per load of 200,000 pounds. Last reading taken 45 hours after preceding measurement.

The nickel-steel jacket used contains about 5 per cent of nickel; the elastic constants of nickel-steel containing $5\frac{1}{2}$ per cent of nickel are taken to be $\mu'=10.8\times 10^6$ pounds per sq. in., $\sigma'=.327$,[1] values not very different from these ordinarily given for steel.

The constants for the Solenhofen limestone of which the rock specimen is made have not been determined; the rock in question resembles in its structure and properties black Belgian marble for which the elastic constants are $\mu=4.33\times 10^6$ pounds per sq. in., $\sigma=.278$.[2]

In calculating the value of β from (11) we may neglect a^2/b^2 and since $b^2/c^2=$ 04 we find $\beta=.77$. Equation (16) then gives

$$\frac{U'}{c}=\frac{P(\text{pound per sq. in.})}{1.57\times 10^9} \tag{18}$$

Taking P = 200,000 pounds per sq. in. we have $U'/c=127\times 10^{-6}$.

The observed radial strain over the center of the specimen is considerably greater than this. We may attribute the difference to a diminution of rigidity. Taking as an extreme case $a=0$ and $\mu=0$,

[1] Mercadier, *Comptes Rendus*, 113, 1891.

[2] Adams and Coker, *Elastic Constants of Rocks*, Carnegie Institution, 1906.

$\sigma=\frac{1}{2}$ (incompressible fluid in the nickel-steel jacket), we have $\beta=0$. Then

$$\left(\frac{U'}{c}\right)_{\mu=0}=\frac{P(\text{pounds per sq. in.})}{.345\times10^{9}}, \qquad (19)$$

which gives for $P=200{,}000$ pounds,

$$\left(\frac{U'}{c}\right)_{\mu=0}=578\times10^{-6}$$

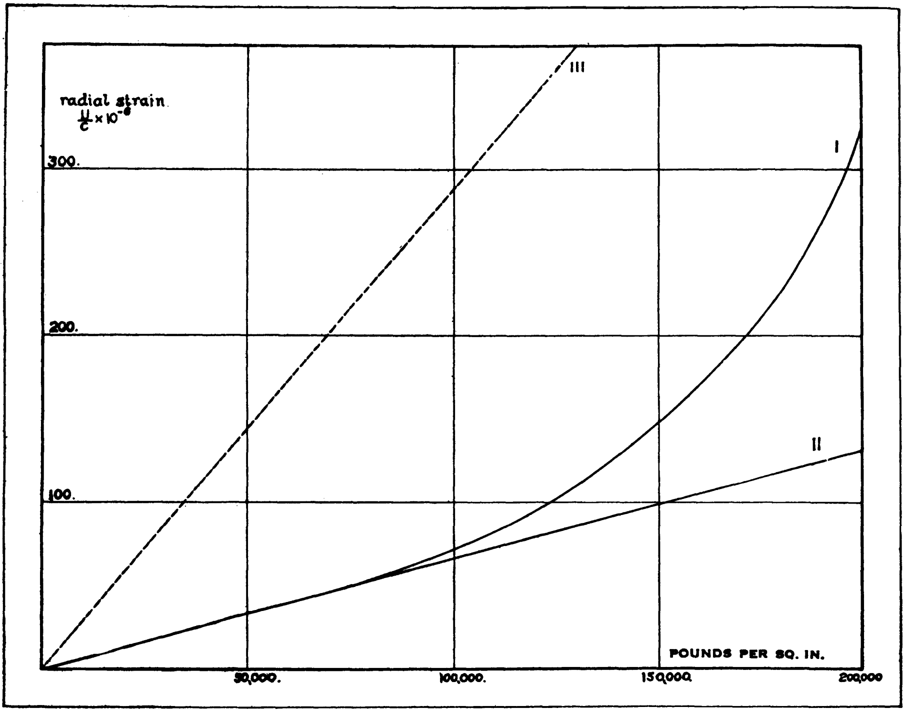


FIG. 2.—Showing relation between radial strain and load for a nickel-steel jacket containing a specimen of Solenhofen limestone under test.

Curve I shows the observed radial strain over the center of the specimen.

Curve II shows the radial strain calculated according to the elastic solid theory for rigidity $\mu=4.33\times10^{6}$ pound per sq. in. and Poisson's ratio $\sigma=.278$.

Curve III shows the calculated relation between strain and stress for a specimen of zero rigidity.

The results given in Table I are shown graphically in Fig. 2, where the radial strains are plotted on a load base. Equation (16) shows that if the rock specimen had behaved throughout as a perfectly elastic solid, U'/c should be proportional to the load, and the curve connecting load and strain should be a straight line. Even taking into account the equatorial bulge of the nickel-steel jacket the proportionality of strain to stress should be exact on the elastic solid theory.[1] We may interpret the results just obtained to a gradual change in the value of the elastic constants which may be regarded as continuous functions of the stress-differences; in fact, if the departure of Curve I of Fig. 2 from a straight line be attributed to a change in the modulus of rigidity, the variation of the latter with stress-difference might be deduced from an analysis of the curve. The theoretical curve for the radial strain over the center of the specimen agrees well with that observed until a load of 80,000 pounds per sq. in. is reached. A curve for the displacement that would be given if the specimen possessed zero rigidity and a Poisson's ratio of $\frac{1}{2}$ is also shown in the diagram. The departure of the observed curve from that obtained on the hypothesis of perfect elasticity shows that the rigidity modulus diminishes under load and that Poisson's ratio increases toward the limit $\frac{1}{2}$. The observations also indicate that when the load is removed the specimen has received a permanent set with a tendency to recovery with time, showing an indication of elastico-viscous properties.[2] The elastic moduli are also permanently changed; a curious fact explained by this means is that the specimen after having been tested shows a smaller diameter across the center than across the ends, whereas the radial displacement was greatest at that point. Since the displacements were greatest in the central part of the specimen it is reasonable to suppose that the rigidity is permanently altered to a lesser value than in the ends. The result is that when the load is removed the lateral pressure exerted by the nickel-steel jacket is able to produce a greater displacement at the center of the specimen than at the ends, so that a slight waist is produced in the rock cylinder.

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[1] Cf. Filon, *loc. cit.*

[2] G. H. Darwin, "On the Bodily Tides of Viscous and Semi-elastic Spheroids," *Phil. Trans. Roy. Soc.*, CLXX, 17.

§ 5. ON THE CONDITIONS OF RUPTURE IN CYCLINDRICAL ROCK SPECIMENS UNDER CONDITIONS OF EXTREME PRESSURE

An examination of equations (13) shows that the stress-differences (i) and (iii) take their maximum values at $r=a$, and that (iii) takes its maximum value at $r=b$. We thus obtain for the maximum values of the stress-differences

$$\left.\begin{aligned}&\text{(i)}\ \widehat{rr}-\widehat{zz}=P\\&\text{(ii)}\ \widehat{rr}-\widehat{\theta\theta}=\frac{\sigma}{1-\sigma}\cdot\frac{2}{1+\beta}\cdot P\\&\text{(iii)}\ \widehat{\theta\theta}-\widehat{zz}=P\left\{1-\frac{\sigma}{1-\sigma}\cdot\frac{1}{1+\beta}\left(1+\frac{a^2}{b^2}\right)\right\}\end{aligned}\right\}\quad(20)$$

We note that (i) > (ii) provided $\beta>\frac{3\sigma-1}{1-\sigma}$. Also (i) > (iii) in all cases except when $\beta=\infty$ $(b=c)$ in which case (i) = (iii).

Thus if

$$\beta>\frac{3\sigma-1}{1-\sigma}\quad\text{(i)} > \text{(ii) and (i)} > \text{(iii)}\qquad(21)$$

so that the specimens will develop the system of surfaces of shear shown in (i) Fig. 1.

$$\text{If } \beta<\frac{3\sigma-1}{1-\sigma}\text{ we have (ii)} > \text{(i)} > \text{(iii)}\qquad(22)$$

In this case the system of surfaces of shear described in (ii), Fig. 1, will make their appearance.

In applying the criterion

$$\beta\gtrless\frac{3\sigma-1}{1-\sigma}\qquad(23)$$

it is necessary to employ instantaneous values of the elastic constants, since from the previous section these are seen to depend on the load. If from (11) we write approximately $\beta=\frac{\mu}{\mu'}\,\frac{1+\sigma}{1-\sigma}$ corresponding to $c=\infty$, and $a=0$, the criterion (23) may be written

$$\mu/\mu'\gtrless\frac{3\sigma-1}{1+\sigma}.$$

As long as $\sigma<\frac{1}{3}$ the first of these criteria given in (21) is always satisfied. We have seen, however, that the rigidity μ tends to diminish with load, while σ tends to the value $\frac{1}{2}$. Under these conditions it is quite possible that the condition $\beta<\frac{3\sigma-1}{1-\sigma}$ given in (22) may be satisfied before the specimen actually begins to rupture in the neighborhood of the cylindrical cavity. The breakdown of a rock specimen under these conditions would be characterized by a tendency to split across a plane through the axis: over a cross-section a system of spiral cracks cutting the cylindrical cavity at an angle would make their appearance and long splinters parallel to the axis would break off from the interior of the cylindrical cavity. Actual cases of fractures of this kind are described by Dr. Adams in the preceding paper.[1] We note that for a solid cylinder ($a=0$) the stress-difference (i) = (iii), so that the conditions for the appearance of the surfaces of shear (i) and (iii) are identical. System (iii) is characterized by the appearance of Luder's lines over the surface of the specimen; these in fact are often observed.

§ 6. ON THE LIMITING STRESS-DIFFERENCE IN THE NEIGHBORHOOD OF A SMALL CYLINDRICAL CAVITY

The preceding analysis may be applied to a determination of the limiting stress-difference S in the neighborhood of the small cylindrical cavities in the specimens of Westerly (Rhode Island) granite tested by Dr. Adams and described in the preceding paper. This granite gave a crushing strength of 27,370 pounds per sq. in. for a two-inch cube. If we note that for this granite $\sigma=.219$,[2] the condition $\beta>\frac{3\sigma-1}{1-\sigma}$ is satisfied and the greatest stress-difference in the specimen is P, so that $S=P$. Experiments 373 and 357 described in Dr. Adams' paper (p. 113) enable a limit to be assigned to the value of S. In experiment 373 a pressure P of 160,000 pounds per sq. in. applied during 75 days produced no change in the diameter of the cylindrical cavity, indicating that no flow or permanent set had taken place. In the case of a similar specimen tested under a pressure P of 200,000 pounds for the same length of time, it was found that the vertical cavity was partly closed up by rock powder. We are therefore able to assert that for a state of stress at ordinary temperature and lasting for 75 days, S lies between the limits of 160,000 and 200,000 pounds per sq. in., i.e.,

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[1] Adams, p. 109 of this journal, Plate I, figs. *c* and *d;* Plate II, fig. *c*.

[2] Adams and Coker, *loc. cit.*

$$160{,}000 < S < 200{,}000 \text{ pounds per sq. in.} \tag{24}$$

Even at a temperature of 550° C. experiment 381 (p. 115) shows that S is greater than 96,000 pounds per sq. in.

We cannot, however, safely assert that flow or rupture would not take place for a stress-difference less than S if the boundary conditions were very different from these already described at the surface of a cylindrical cavity. In order to settle this point experiments of the type described in § 4 would be required. The value given in (24) for S is very much greater than the limiting stress-difference of 27,000 pounds per sq. in. obtained by means of the usual crushing test. Since this granite is typical of the great mass of igneous rocks which make up the earth's crust, it follows that in geophysical problems which involve a knowledge of limiting stress-difference in the earth's crust, a value considerably higher than that usually employed must be taken.

§ 7. ON THE EXISTENCE OF CAVITIES IN MEDIA UNDER STRESS

(*i*) *State of Stress in the Neighborhood of Cavities in Elastic Media under Shear*

It can be shown that in the neighborhood of both a spherical and a cylindrical cavity, the shear can be nearly equal to twice the shear at a distance.[1] We see therefore that the medium near a cavity will collapse if the stress-difference at a distance exceeds half the limiting stress-difference of the rock material. If the cavity be small this limiting stress-difference may be taken to be the value of S given in equation (24).

(*ii*) *State of Stress in the Neighborhood of a Spherical Cavity in a Medium under Pressure*

If p be the hydrostatic pressure in the medium it can easily be shown from the solution given by Love[2] for a spherical shell under external pressure that the maximum stress-difference in the neighborhood of the cavity is $\frac{3}{2}p$. This result is independent of the elastic constants and of the radius of the cavity. If then the pressure in the medium exceeds $\frac{2}{3}$ the limiting stress-difference of the rock, the cavity will break down. The value of S given in (24) for the limiting stress-differences may be employed provided the radius of the cavity be small.

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[1] Love, *Elasticity*, 2d ed., 245. See also a paper by Larmor, *Phil. Mag.* (Ser. 5), XXXIII, 1892, p. 77.

[2] Love, *op. cit.*, 139.

(*iii*) *On the Stability of a Spherical Cavity in a Medium under Pressure*

That the result quoted in (ii) should be independent of the radius of the cavity seems contrary to experience. The explanation of this anomaly lies in the fact that the question turns on an examination of *stability* rather than on a purely *static* consideration of stress. A problem of a similar kind is that of the collapse of a thin-walled cylinder (e.g. a boiler flue) under external pressure.[1] The result shows that for a cylindrical shell of infinite length collapse will not occur as long as the pressure does not exceed

$$p=\frac{4\mu}{1-\sigma}\left(\frac{t}{a}\right)^3 \tag{25}$$

where μ and σ denote the rigidity modulus and Poisson's ratio for the material and t/a is the ratio of the thickness to the diameter. The vibrations in the neighborhood of a spherical cavity in an infinite medium under hydrostatic pressure do not appear to have been explicitly worked out; the problem would correspond to the external solution of the problem of the vibrations of a solid elastic sphere.[2] In so far as the stability depends on the radial vibrations of the cavity, a solution could easily be adapted from Poisson's solution for the case of a solid elastic sphere.[3] The result of an investigation on stability would give a result analogous to (25) for a cylindrical shell. When the pressure p is specified the condition of stability will lead to a limiting radius which will be determined by the elastic constants of the medium. A cavity having a greater radius than this limiting radius will be unstable and collapse in such a way as to form a number of smaller cavities each of which might be stable.

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[1] Love, *op. cit.*, 530.

[2] Cf. Love, "Gravitational Stability of the Earth," *Phil. Trans.*, 207 A, 1908.

[3] Poisson, "Mémoire sur l'équilibre et le mouvement des corps élastiques," *Mém. Paris Acad.*, t. 8, 1829.

§ 8. NOTE ON CONDITIONS OF STRESS IN THE EARTH'S CRUST

The determination of conditions of stress existing in the earth's crust constitutes one of the most difficult problems of geodynamics and one on which many recent investigations have been published. If the earth is considered as a perfect sphere in which the density is a function of the radius only, the problem can be solved without difficulty in terms of the elastic solid theory.[1] The application of these results throughout the interior of the earth is not legitimate because an infinite number of laws of density can be formulated to represent the known distribution of mass throughout the earth. Near the surface considerations of surface inequalities vitiate the result. In this connection Love states,[2] "Apart from the difficulty concerning the initial stress in a gravitating body of the size of the earth, a difficulty which we seem unable to avoid without treating the material as incompressible, there is another difficulty in the application of such an analysis to problems concerning compressible gravitating bodies. In the analysis we take account of the attraction of the inequality at the surface, but we neglect the inequalities of the internal attraction which arise from the changes of density in the interior; yet these inequalities of attraction are of the same order of magnitude as the attraction of the surface inequality."

In a recent work Love[3] has attacked the problem of isostatic support of continents and mountains. "The problem admits of an infinite number of solutions even if the distribution of the mass is known. . . . . The problem must remain indeterminate, and all we can do is to obtain explicitly one or more of the infinitely numerous solutions. One solution of the problem was obtained by Sir G. Darwin[1] by assuming that the stress is connected with a displacement by the same equations as hold in the ordinary theory of an elastic *incompressible* solid, ... "

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[1] Lamé solves the problem for an isotropic elastic sphere under its own gravitation (Love, *Elasticity*, 2d ed., 140); the result can only be reconciled with fact by the assumption of incompressibility or by taking into account a state of initial stress. L. M. Hoskins in a paper "Flow and Fracture of Rocks as Related to Structure" (*U.S. Geological Survey*, 1894–95, Part I, 854), solves the problem throughout a compressible concentric crust of uniform density and small thickness; no account is taken, however, of surface inequalities or of considerations of isostasy.

[2] Love, *Elasticity*, 2d ed., 253.

[3] Love, *Some Problems of Geodynamics* (*Adams Prize Essay*, 1911), chaps. ii and iii, p. 12.

The theoretical developments turn on the calculation of stress-differences corresponding to surface inequalities representing continental blocks and mountains. Expressions for the hydrostatic pressure at any depth are not explicitly worked out although it would be possible to calculate from the analysis given the pressure at any depth due to those harmonic inequalities which Love has shown can be made to represent the existing distribution of land and water and the general form of continental blocks. For most purposes it is sufficient to calculate the pressure at any depth as that due to the weight of the corresponding column of rock: this formula gives a value of greater relative accuracy the greater the depth at which the pressure is calculated.

§ 9. DEPTH AT WHICH CAVITIES CAN EXIST

(*i*) *Collapse Due to Stress-Differences in the Earth's Crust*

Love[2] shows that on the isostatic theory the maximum stress-difference due to a harmonic inequality of the third order representing a continental block of maximum elevation 2 kms. is approximately the weight of a column of rock equal to 0.0208 of this greatest height, i.e., about 0.0114 of a metric tonne per sq. cm. (162 pounds per sq. in.). This value is greater than that brought about by inequalities of the first and second order. The stress-difference is also worked out for a parallel series of mountain ranges 400 kilometers apart and having a height of 4 kms. from crest to valley-bottom; it is shown that the greatest value of the stress-difference is about .26 metric tonnes per sq. cm. (3,700 pounds per sq. in.). In both cases the stress-difference is very much less than $\frac{1}{2} S$ so that no state of stress-difference due to weights of continents or mountains is intense enough to cause the material of the earth's crust to rupture in the neighborhood of a small cavity.

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[1] Darwin, "On the Stresses Caused in the Interior of the Earth by the Weight of Continents and Mountains," *Phil. Trans. Roy. Soc.*, CLXXIII (1882).

[2] Love, *op. cit.*, 36; also 47.

No account is here taken of stress-difference brought about by the folding of the earth's crust or that due to residual effects of a state of very intense stress which may have been brought into existence at a very remote period of the earth's history.

(*ii*) *Collapse Due to Pressure in the Earth's Crust*

We have noticed in § 8 that the pressure at any point of the earth's crust may be roughly represented by the weight of a column of rock of that depth. If w represent the average weight of rock at the earth's surface per cubic cm. and h the depth in cms., a small spherical cavity can exist provided

$$wh<\tfrac{2}{3}S \tag{26}$$

where S is the limiting stress-difference of § 6 expressed in grammes per sq. cm. Making use of the numerical values corresponding to a limiting stress-difference between 160,000 and 200,000 pounds per sq. in. (1.13×10^6 and 1.41×10^6 grammes per sq. cm.) for Westerly granite and taking 2.7 as the mean density of surface rocks, we can assert that small cavities in the earth's crust will not collapse provided the depth does not exceed a quantity h whose value lies between 27.7 and 33.7 kilometers (h between 17.2 and 20.9 miles). This estimate is several times greater than that deduced by Hoskins,[1] the reason being that it is necessary to make use of a very much higher value of limiting stress-difference than that employed in the investigation referred to. Even at a temperature of 550° C. which is supposed to exist at a point 11 miles below the earth's surface, Adams has shown that S is greater than 95,000 pounds per sq. in. which corresponds to a depth of 15 miles. Any internal pressure in the cavity (due to water-pressure, steam, etc.) would increase the above estimate for the depths at which cavities can exist.

The *size* of a cavity which can exist at a given depth depends on considerations of *stability* and would demand a separate investigation.

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[1] Hoskins, *op. cit.*, 859. Hoskins expresses the result of his investigation in the form "cavities must certainly close up if $wh>S$." The value for S is taken to be 25,000 pounds per sq. in. The much higher value between 160,000 and 200,000 pounds per sq. in. indicated by experiments on Westerly granite shows that Hoskins' estimate for the depth at which cavities must close, must be increased between 7 and 8 times.

§ 10. SUMMARY AND CONCLUSIONS

1. Secs. 1 to 3 are devoted to a mathematical discussion of the state of stress existing in cylindrical rock specimens incased in heavy nickel-steel jackets and described by Dr. Adams in an accompanying paper.

2. In sec. 4 it is shown that observations on the radial strains of the nickel-steel jacket agree well with the calculated values as long as the load does not exceed a certain value; beyond that point the results of experiment indicate a change in the elastic constants with load; an analysis of observations of this type could be made to give a relation between the rigidity and the load for stress-differences beyond the elastic limit. The results of sections 4 and 5 also give a good account of the manner in which the rock specimens break down under load.

3. The experimental results of Dr. Adams discussed in sec. 6 of the present paper show that in the neighborhood of a small cylindrical cavity, in a specimen of Westerly granite, no flow or set takes place when the stress-difference amounts to as much as 160,000 pounds per sq. in. The cavity just begins to break down for a stress-difference of 200,000 pounds per sq. in. We therefore assign to the limiting stress-difference in the neighborhood of small cavities in rocks typical of that forming the greater part of the earth's crust, a value S between 160,000 and 200,000 pounds per sq. in. for ordinary temperatures.

4. Secs. 7 to 9 deal with the existence of cavities in media under stress with especial reference to the depth at which cavities in the earth's crust can remain open.

It is shown that no state of shearing stress in the crust of the earth, due to the weights of continents and mountains can cause the collapse of the rock in the neighborhood of a small cavity.

It is also shown that as far as hydrostatic pressure in the earth's crust is concerned a small cavity at ordinary temperatures will remain open provided the depth does not exceed a value between 17.2 and 20.9 miles. At a temperature of 550° C. supposed to exist 11 miles below the earth's surface, cavities will remain open when submitted to considerably greater pressures than are found at this depth. These values greatly exceed previous estimates because experiment shows that a much higher value of limiting stress-difference than that usually employed must be taken in the neighborhood of small cavities.

5. The *size* of a cavity which can exist at a given depth depends on considerations of *stability* and would demand a separate investigation.

January 30, 1912